The singular dynamic method for dynamic contact of thin elastic structures

Cédric Pozzolini, Yves Renard, Michel Salaün

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Cédric Pozzolini [1], Yves Renard [2], Michel Salaün [3]

Abstract

The aim of this paper is to review the use of the singular dynamic method to obtain space semi-discretization of the dynamic impact of thin structures. The principle of these methods is the use of a singular mass matrix obtained by different discretizations of the deflection and velocity. The obtained semi-discretized problem is proved to be well-posed and energy conserving. The method is applied on some membrane, beam and plate models and associated numerical experiments are discussed.

Keywords: thin structures, unilateral contact, finite element methods, variational inequalities.

Introduction

When the discretization of impact of elastic structures is addressed, it is generally noted that the vast majority of traditional time integration schemes show spurious oscillations on the contact displacement and stress (see for instance [11, 7, 8]). Moreover, these oscillations do not disappear when the time step decreases. Conversely, they tend to increase which is a characteristic of order two hyperbolic equations with unilateral constraints that makes it very difficult to build stable numerical schemes. These difficulties have already led to many researches under which a variety of solutions were proposed. Some of them consist in adding damping terms (see [26] for instance), but with a loss of accuracy on the solution, or to implicit the contact stress [5, 4] but with a loss of kinetic energy which could be independent of the discretization parameters (see the numerical experiments). Some energy conserving schemes have also been proposed in [10, 27, 16, 15, 7, 8]. Unfortunately, these schemes, although more satisfactory than most of the other ones, lead to large oscillations on the contact stress. Besides, most of them do not strictly respect the constraint.

This paper focuses on a class of methods introduced for impact problems in [24] whose principle is to make different approximations of the solution and of its time derivative. Such a principle was already studied for linear elastodynamics in [9]. Compared to the classical space semi-discretization, this corresponds to a singular modification of the mass matrix. In this sense, it is in the same class of methods than the mass redistribution method proposed in [11, 12] for elastodynamic contact problems. The main feature is to provide a well-posed space semi-discretization. The numerical tests show that it has a crucial influence on the stability of standard schemes and on the quality of the approximation, especially for the computation of Lagrange multipliers corresponding to the constraints.

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The aim of this paper is to gather the recent results obtained on the singular dynamic method for different models of thin structures and to present an overview of the main challenges still posed by the discretization of the impact problems of such structures. As in [24], the method is first described on an abstract hyperbolic equation on which the well-posedness of the semi-discretized problem by finite elements is proven. The method is then described on two models of thin elastic structures, namely membrane and Kirchhoff-Love plate models (note that the applciation to Euler-Bernouilli beam model can be in [23]). Some numerical tests for all these models are given and discussed. Finally, we present some perspectives and open problems.

1 The method for an abstract hyperbolic equation

The method is introduced in [24] on the following abstract hyperbolic problem. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $H = L^2(\Omega)$ the standard Hilbert space of square integrable functions on Ω. Let W be a Hilbert space such that $W \subset H \subset W'$, with dense compact and continuous inclusions and let $A : W \to W'$ be a linear self-adjoint elliptic continuous operator. We consider the following problem

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that} \\ \dfrac{\partial^2 u}{\partial t^2}(t) + Au(t) \in f - N_K(u(t)) \quad , \quad \text{for a.e. } t \in \ (0,T] \quad , \\ u(0) = u_0 \quad , \quad \dfrac{\partial u}{\partial t}(0) = v_0 \quad , \end{cases} \tag{1}$$

where K is a closed convex nonempty subset of W, $f \in W'$, $u_0 \in K$, $v_0 \in H$, $T > 0$ and $N_K(u)$ is the normal cone to K defined by (see for instance [3] for more details)

$$N_K(u) = \begin{cases} \emptyset \quad , \quad \text{if } u \notin K \quad , \\ \{f \in W' : \langle f, w-u \rangle_{W',W} \leq 0 \quad , \quad \forall w \in K\} \quad , \quad \text{if } u \in K \quad . \end{cases}$$

This means that $u(t)$ satisfies the second order hyperbolic equation and is constrained to remain in the convex K. There is no general result of existence nor uniqueness for the solution to this problem. Some existence results for a scalar Signorini problem can be found in [17, 14]. Introducing now the linear and bilinear symmetric maps

$$l(v) \ = \ \langle f, v \rangle_{W',W} \quad , \quad a(u,v) \ = \ \langle Au, v \rangle_{W',W} \quad ,$$

Problem (1) can be rewritten as the following variational inequality:

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that for a.e. } t \in \ (0,T] \\ \langle \dfrac{\partial^2 u}{\partial t^2}(t), w - u(t) \rangle_{W',W} + a(u(t), w-u(t)) \geq l(w - u(t)) \quad , \quad \forall w \in K \quad , \\ u(0) = u_0 \quad , \quad \dfrac{\partial u}{\partial t}(0) = v_0 \quad . \end{cases} \tag{2}$$

Note that the terminology "variational inequality" is used here in the sense that Problem (1) derives from the conservation of the energy functional

$$J(t) = \frac{1}{2} \int_\Omega (\frac{\partial u}{\partial t}(t))^2 dx + \frac{1}{2} a(u(t), u(t)) - l(u(t)) + I_K(u(t)) \quad ,$$

where $I_K(u(t))$ is the convex indicator function of K. However, it is generally not possible to prove that each solution to Problem (2) is energy conserving, due to the weak regularity involved.

2 Approximation and well-posedness result

The aim of this section is to present well-posed space semi-discretizations of Problem (2). The adopted strategy is to use a Galerkin method with different approximations of u and of $v = \dfrac{\partial u}{\partial t}$. Let W^h and H^h be two finite dimensional vector subspaces of W and H respectively. Let $K^h \subset W^h$ be a closed convex nonempty approximation of K. The proposed approximation of Problem (2) is the following mixed approximation:

$$\left\{\begin{array}{l}\text{Find } u^h : [0,T] \to K^h \text{ and } v^h : [0,T] \to H^h \text{ such that} \\ \displaystyle\int_\Omega \frac{\partial v^h}{\partial t}(w^h - u^h)dx + a(u^h, w^h - u^h) \geq l(w^h - u^h) \quad , \quad \forall w^h \in K^h \ , \ \forall t \in \ (0,T] \ , \\ \displaystyle\int_\Omega (v^h - \frac{\partial u^h}{\partial t})q^h dx = 0 \quad , \quad \forall q^h \in H^h \quad , \quad \forall t \in \ (0,T] \quad , \\ u^h(0) = u_0^h \quad , \quad v^h(0) = v_0^h \quad , \end{array}\right. \tag{3}$$

where $u_0^h \in K^h$ and $v_0^h \in H^h$ are some approximations of u_0 and v_0 respectively. Of course, when $H^h = W^h$, this corresponds to a standard Galerkin approximation of Problem (2).

Let φ_i , $1 \leq i \leq N_W$, and ψ_i , $1 \leq i \leq N_H$, be some basis of W^h and H^h respectively, and let the matrices A, B and C, of sizes $N_W \times N_W$, $N_H \times N_W$ and $N_H \times N_H$ respectively, and the vectors L, U and V, of size N_W , N_W and N_H respectively, be defined by

$$A_{i,j} = a(\varphi_i, \varphi_j) \quad , \quad B_{i,j} = \int_\Omega \psi_i \varphi_j dx \quad , \quad C_{i,j} = \int_\Omega \psi_i \psi_j dx \quad ,$$

$$L_i = l(\varphi_i) \quad , \quad u^h = \sum_{i=1}^{N_W} U_i \varphi_i \quad , \quad v^h = \sum_{i=1}^{N_H} V_i \psi_i \quad .$$

Then, U and V are linked by the equation $CV(t) = B\dot{U}(t)$. So V can be eliminated since C is always invertible, which leads to the relation $V(t) = C^{-1}B\dot{U}(t)$. Consequently, Problem (3) can be rewritten as

$$\left\{\begin{array}{l}\text{Find } U : [0,T] \to \overline{K}^h \text{ such that} \\ (W - U(t))^T (M\ddot{U}(t) + AU(t)) \geq (W - U(t))^T L \quad , \quad \forall W \in \overline{K}^h \quad , \quad \forall t \in \ (0,T] \quad , \\ U(0) = U_0 \quad , \quad B\dot{U}(0) = CV_0 \quad . \end{array}\right. \tag{4}$$

In comparison with the standard approximation where $H^h = W^h$, the only difference introduced by the presented method is to replace the standard mass matrix $\left(\displaystyle\int_\Omega \varphi_i \varphi_j dx\right)_{i,j}$ by $M = B^T C^{-1} B$. In the interesting cases where $\dim(H^h) < \dim(W^h)$, it corresponds to replace the standard invertible mass matrix by a singular one.

Although the analysis could probably be extended to more complex situations, we assume that K^h is defined by a finite number of linear constraints as

$$K^h = \{w^h \in W^h : g^i(w^h) \leq \alpha^i \ , \ 1 \leq i \leq N_g\} \quad ,$$

where $\alpha^i \in \mathbb{R}$ and $g^i : W^h \to \mathbb{R}$, $1 \leq i \leq N_g$, are some linearly independent linear maps. Of course, this restricts the possibilities concerning the convex K since K^h is supposed to be an approximation of K. With vector notations, this leads to

$$\overline{K}^h = \{W \in \mathbb{R}^{N_W} : (G^i)^T W \leq \alpha_i \ , \ 1 \leq i \leq N_g\} \quad ,$$

where $G^i \in \mathbb{R}^{N_W}$ are such that $g^i(w^h) = (G^i)^T W$, $1 \leq i \leq N_g$. We will also denote by G the $N_W \times N_g$ matrix whose components are

$$G_{ij} = (G^i)_j \quad .$$

Let us consider the subspace F^h of W^h defined by

$$F^h = \left\{ w^h \in W^h : \int_\Omega w^h q^h = 0 \ , \ \forall q^h \in H^h \right\} \quad .$$

Then, the corresponding set $F = \left\{ W \in \mathbb{R}^{N_W} : \sum_{i=1}^{N_W} W_i \varphi_i \in F^h \right\}$ is such that $F = \text{Ker}(B)$. In this framework, we consider the following condition:

$$\inf_{\substack{Q \in \mathbb{R}^{N_g} \\ Q \neq 0}} \sup_{\substack{W \in F \\ W \neq 0}} \frac{Q^T G W}{||Q|| \ ||W||} \ > \ 0 \quad , \tag{5}$$

where $||Q||$ and$||W||$ stand for the Euclidean norm of Q in $\mathbb{R}^{N_g}$ and W in $\mathbb{R}^{N_W}$ respectively. This condition is equivalent to the fact that the linear maps g^i are independent on F^h and also to the fact that G is surjective on F. A direct consequence is that it implies $\dim(F^h) \geq N_g$ and consequently

$$\dim(H^h) \ \leq \ \dim(W^h) \ - \ N_g \quad .$$

This again prescribes some conditions on the approximations which link W^h, H^h and also K^h. We will see in Section 3 that this condition can be satisfied for interesting practical situations. We can now prove the following result:

Theorem 1 *If W^h, H^h and K^h satisfy condition (5), then Problem (4) admits a unique solution. Moreover, this solution is Lipschitz-continuous with respect to t.*

Proof of this theorem can be found in [24]. In particular, it is based on the following result allowing a decomposition of the solution:

Lemma 1 *If W^h, H^h and K^h satisfy condition (5), then there exists a sub-space of $\mathbb{R}^{N_W}$, say F^c, such that $F^c \subset Ker(G)$ and such that F and F^c are complementary sub-spaces.*

Moreover, the following energy conservation is proved:

Theorem 2 *If W^h, H^h and K^h satisfy condition (5), then the solution $U(t)$ to Problem (4) is energy conserving in the sense that the discrete energy*

$$J^h(t) = \frac{1}{2} \dot{U}^T(t) M \dot{U}(t) + \frac{1}{2} U^T(t) A U(t) - U^T(t) L \quad ,$$

is constant with respect to t.

3 Application to a membrane model

This section provides a simple but interesting situation for which some consistent approximations satisfy the condition (5). When $W = H^1(\Omega)$ and $K = \{w \in W : w \geq 0 \text{ a.e. on } \Omega\}$, we consider the following problem

$$\begin{cases} \text{Find } u : [0,T] \to K \text{ such that} \\ \dfrac{\partial^2 u}{\partial t^2}(t) - \Delta u(t) \in f - N_K(u(t)) \quad \text{in } \Omega \quad , \quad \text{for a.e. } t \in \ (0,T] \quad , \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N \quad , \\ u = 0 \quad \text{on } \Gamma_D \quad , \\ u(0) = u_0 \quad , \quad \dfrac{\partial u}{\partial t}(0) = v_0 \quad , \end{cases}$$

where Γ_N and Γ_D is a partition of $\partial\Omega$, Γ_D being of non zero measure in $\partial\Omega$. This models for instance the contact between an antiplane elastic structure with a rigid foundation or a stretched drum membrane under an obstacle condition. In this situation, the mass redistribution method presented in [12] is not usable since the area subjected to potential contact is the whole domain. Consequently, this method would lead to suppress the mass on the whole domain which is a non consistent drastic change of the problem.

We build now the approximation spaces thanks to finite element method. Let $\mathscr{T}^h$ a regular triangular mesh of Ω (in the sense of Ciarlet [2], h being the diameter of the largest element) and W^h be the following P_1+ finite element space

$$W^h = \left\{ w^h \in \mathscr{C}^0(\Omega) : w^h = \sum_{a_i \in \mathscr{A}} w_i \varphi_i + \sum_{T \in \mathscr{T}^h} w_T \varphi_T \right\} \quad ,$$

where $\mathscr{A}$ is the set of the vertices of the mesh which do not lie on Γ_D . Then, φ_i, $a_i \in \mathscr{A}$, are the piecewise linear functions satisfying $\varphi_i(a_j) = \delta_{ij}$, where δ_{ij} is Kronecker symbol, i.e. the shape functions of a P_1 Lagrange finite element method on $\mathscr{T}^h$. Each function φ_T, $T \in \mathscr{T}^h$, is the cubic bubble function whose support is T. Let H^h be the P_0 finite element space

$$H^h = \left\{ v^h \in L^2(\Omega) : v^h = \sum_{T \in \mathscr{T}^h} v_T \mathbb{1}_T \right\} \quad ,$$

and, finally, let K^h be defined as

$$K^h = \left\{ w^h \in W^h : w^h(a_i) \geq 0 \quad , \quad \text{for all } a_i \in \mathscr{A} \right\} \quad , \tag{6}$$

which means that the constraints are only prescribed at the vertices of the mesh. Then, it is proved in [24] that this choice of W^h, H^h and K^h satisfies condition (5).

4 Extension to a plate impacting a rigid obstacle

Let us consider a thin elastic plate. For this kind of structures, starting from *a priori* hypotheses on the expression of the displacement fields, a two-dimensional problem is usually derived from the three-dimensional elasticity formulation by means of integration along the thickness, say $2\ \varepsilon$. For the Kirchhoff-Love plate model, the only variable is the normal deflection, say $u(x,t)$, and is set down on the mid-plane of the plate Ω. So the Kirchhoff-Love elastodynamical model reads as

$$\left\{\begin{array}{l} \text{Find } u \;=\; u(x,t) \text{ with } (x,t) \;\in\; \Omega \;\times\; (0,T] \text{ such that for any } w \in W \\ \\ \displaystyle\int_\Omega 2\rho\varepsilon \;\frac{\partial^2 u}{\partial t^2}\; w\ dx \;+\; a(u,w) \;=\; \int_\Omega f\ w\ dx \quad , \end{array}\right.$$

where

$$a(u,w) \;=\; \int_\Omega \frac{2\ E\ \varepsilon^3}{3\ (1-\nu^2)} \;\left[(1-\nu)\;\frac{\partial^2 u}{\partial x_\alpha\ \partial x_\beta} \;+\; \nu\ \Delta u\ \delta_{\alpha\beta}\right]\; \frac{\partial^2 w}{\partial x_\alpha\ \partial x_\beta}\ dx \quad ,$$

where the mechanical constants, for a plate made of a homogeneous and isotropic material, are its Young modulus E, its Poisson ratio ν and its mass density ρ. Moreover, $\delta_{\alpha\beta}$ is the Kronecker symbol and the summation convention over repeated indices is adopted, Greek indices varying in $\{1,2\}$. If the plate is assumed to be clamped on a non-zero measure part of the boundary $\partial\Omega$ denoted Γ_c and free on Γ_f , such as $\partial\Omega \;=\; \Gamma_c \;\cup\; \Gamma_f$, the space of admissible displacements is

$$W \;=\; \{\ w \in H^2(\Omega)\ /\ w(x) \;=\; 0 \;=\; \partial_n w(x)\ ,\ \forall x \;\in\; \Gamma_c\ \} \quad ,$$

where $\partial_n w$ is the normal derivative along Γ_c . Finally, the associated initial conditions are

$$u(x,0) \;=\; u_0(x) \quad , \quad \frac{\partial u}{\partial t}(x,0) \;=\; v_0(x) \quad , \quad \forall x \;\in\; \Omega \quad .$$

Let us now introduce the dynamic frictionless Kirchhoff-Love equation with Signorini contact conditions along the plate. We assume that the plate motion is also limited by rigid obstacles located above and below the plate. So, the displacement is constrained to belong to the convex set

$$K \;=\; \{w \in W \;:\; g_1(x) \;\le\; w(x) \;\le\; g_2(x) \quad , \quad \forall x \in \Omega\} \quad ,$$

where g_1 and g_2 are two maps which still satisfy $g_1(x) < 0 < g_2(x)$, for all $x \in \Omega$. Then, the mechanical frictionless elastodynamic problem for a plate between two rigid obstacles can be written as the following variational inequality

$$\left\{\begin{array}{l} \text{Find } u : [0,T] \to K \text{ and } v : [0,T] \to L^2(\Omega) \text{ such for a.e. } t \in (0,T] \\ \displaystyle\int_\Omega 2\rho\varepsilon\ \frac{\partial v}{\partial t}\ (w-u)\ dx \;+\; a(u,w-u) \;\ge\; \int_\Omega f\ (w-u)\ dx \quad , \quad \forall w \in K \quad , \\ \displaystyle\int_\Omega (v - \frac{\partial u}{\partial t})\ q\ dx \;=\; 0 \quad , \quad \forall q \in L^2(\Omega) \quad , \\ \displaystyle u(x,0) \;=\; u_0(x) \in K \quad , \quad \frac{\partial u}{\partial t}(x,0) \;=\; v_0(x) \quad , \quad \forall x \in \Omega \quad . \end{array}\right.$$

Let us now introduce the space discretization of the displacement. As the Kirchhoff-Love model corresponds to a fourth order partial differential equation, a conformal finite element approximation needs the use of continuously differentiable elements. Among such ones (see [2]), the reduced HCT (Hsieh-Clough-Tocher) triangles, Argyris triangles, and FVS (Fraeijs de Veubeke-Sanders) quadrangles are of particular interest. For the HCT (resp. FVS) element, the triangle (resp. quadrangle) is divided into three (resp. four) sub-triangles. The basis functions of these elements are P_3 polynomials on each sub-triangle and matched

$\mathscr{C}^1$ across each internal edge. In addition, to decrease the number of degrees of freedom, the normal derivative is assumed to vary linearly along the external edges of the elements. Finally, both for HCT triangles and quadrangles, there are only three degrees of freedom on each node: The value of the function and its first derivatives. The Argyris triangle use a complete polynomial of degree five. The degrees of freedom consist of function values and first and second derivatives at the vertices in addition to normal derivatives at the midpoints of the sides.

In [**?**], such elements for the deflection, piecewise constant velocity and still a nodal contact condition on each vertex of the mesh are numerically shown to satisfy the inf-sup condition (5).

Remark 1 *Since we deal with a fourth order problem with respect to the space derivative, it is not possible to consider a linear space approximation. In fact, for this plate model, we use the classical HCT, Argyris or FVS elements to approximate the numerical displacement. In the above approximation of K, as we consider only constraints on node displacements, the effect of the derivatives, namely the curvature, is not taken into account. Then, in this framework, the plate could cross the obstacle between two nodes, but we shall neglect this aspect in the following.*

5 Numerical discussion

5.1 Midpoint scheme

As far as numerical results are concerned, in this paper, we mainly use a midpoint scheme for the time discretization of the problem. It is an interesting scheme since it is energy conserving on the linear part (equation without constraint) but, of course, any other stable scheme can be applied. For exemple, in [23], Newmark schemes are also used. So, if Δt stands for the time step, the midpoint scheme, applied on all the previous problems, consists in finding $U^{n+1/2}$ in K^h such that

$$\left\{\begin{array}{l} (W - U^{n+1/2})^T \; (MZ^{n+1/2} + AU^{n+1/2}) \;\geq\; (W - U^{n+1/2})^T \; F^n \quad , \quad \forall W \in K^h \quad , \\[2ex] U^{n+1/2} \;=\; \dfrac{U^n + U^{n+1}}{2} \quad , \quad V^{n+1/2} \;=\; \dfrac{V^n + V^{n+1}}{2} \quad , \\[2ex] BU^{n+1} \;=\; BU^n + \Delta t C V^{n+1/2} \quad , \quad CV^{n+1} \;=\; CV^n + \Delta t B Z^{n+1/2} \quad , \end{array}\right. \tag{7}$$

where M and A are the mass and the rigidity matrices corresponding to each discretized problem, while $Z^{n+1/2}$ is the acceleration at "middle time step" $n + 1/2$ and V^k an approximation of the velocity at time $k\Delta t$. Moreover, $Z^{n+1/2}$ can be eliminated then, a new formulation of (7) is

$$\left\{\begin{array}{l} U^n \text{ and } V^n \text{ being given, find } U^{n+1/2} \in K^h \text{ such that} \\[2ex] (W - U^{n+1/2})^T \; (\dfrac{4}{\Delta t^2} M U^{n+1/2} + AU^{n+1/2}) \;\geq\; (W - U^{n+1/2})^T \; \bar{F}^n \quad , \quad \forall W \in K^h \quad , \\[2ex] \text{where } \bar{F}^n \;=\; F^n + \dfrac{4}{\Delta t^2} M \; U^n + \dfrac{2}{\Delta t} B^T \; V^n \\[2ex] U^{n+1} \;=\; 2U^{n+1/2} - U^n \quad , \quad V^{n+1} \;=\; 2C^{-1} \; B \; \dfrac{U^{n+1} - U^n}{\Delta t} \; - \; V^n \quad . \end{array}\right.$$

Let us note that this finite-dimensional variational inequalities has always a unique solution even if M is singular.

5.2 Case of the membrane model

We present now some numerical experiments on the membrane problem, with

$$\Omega = (0,1) \ \times \ (0,1) \quad , \quad \Gamma_D = \partial\Omega \quad , \quad \Gamma_N = \emptyset \quad , \quad f = -0.6 \quad .$$

The initial condition is $u(x,0) = 0.02$, $\dfrac{\partial u}{\partial t}(x,0) \;=\; 0$, for all $x \in \Omega$, and we consider a non-homogeneous Dirichlet condition $u(x,t) = 0.02$, for all $x \in \partial\Omega$.

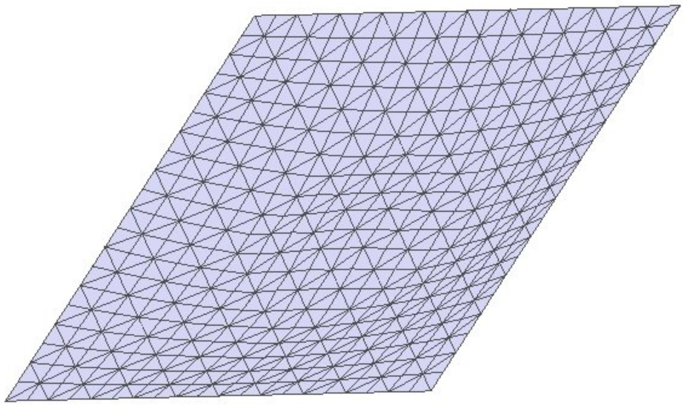

Figure 1: *A mesh with $h = 0.05$.*

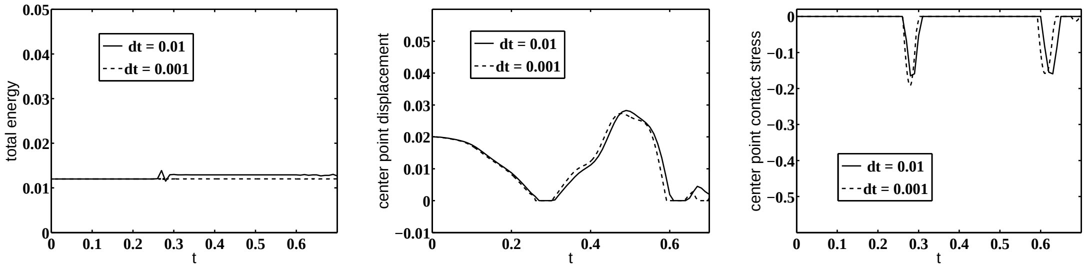

Figure 2: *Energy evolution (left), Displacement at the center point* $(0.5, 0.5)$ *(center) and Contact stress at the center point (right) - P_1+/P_0 method with a midpoint scheme and $h = 0.1$.*

The mesh, we used, is structured and can be viewed on Figure 1, where the solution is represented during the first impact on the obstacle. The numerical experiments are performed with our finite element library Getfem++ [25]. A semi-smooth Newton method is used to solve the discrete problem (see [1, 13]). All the numerical experiments use the same definition of convex K^h, given by (6).

The first numerical test is made with the midpoint scheme and the approximation presented in Section 3, that is a P_1+/P_0 method (P_1+ for displacement and P_0 for velocity).

In good accordance with the theoretical results, the curves on Figure 2 show that the energy tends to be conserved when the time step decreases (an experiment with $\Delta t = 10^{-4}$ has been performed but the difference with the one for $\Delta t = 10^{-3}$ is not visible). Moreover, both the displacement and the contact stress, taken at the point $(0.5, 0.5)$, are smooth and converge satisfactorily when the time step decreases.

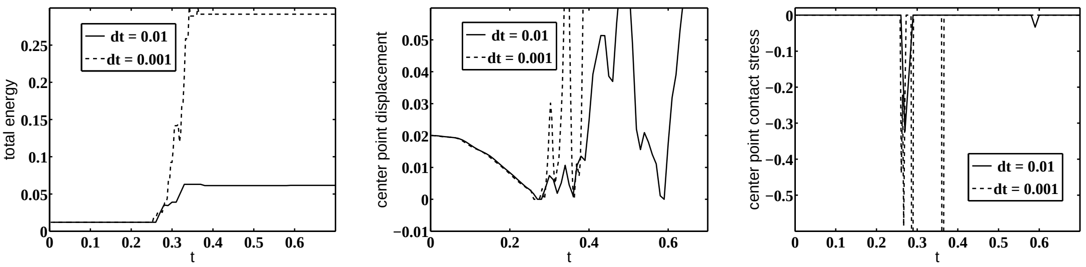

Figure 3: *Energy evolution (left), Displacement at the center point* $(0.5, 0.5)$ *(center) and Contact stress at the center point (right) -* P_1/P_0 *method with a midpoint scheme and* $h = 0.1$.

Conversely, the curves on Figure 3, obtained for a P_1/P_0 method, are unstable. The energy is growing very fast after the first impact. The displacement and the contact stress are very oscillating and do not converge. Moreover, the instabilities are more important for the smallest time step. This method does not satisfy the condition (5) since $\dim(H^h) \geq \dim(W^h)$.

An interesting situation is also presented in Figures 4, 5 and 6, where a backward Euler scheme is used. This time integration scheme is unconditionally stable because it is possible to prove that the discrete energy decreases from an iteration to another (see [11] for instance). This is the case for any choice of W^h and H^h. Consequently, this method presents some smooth results for the displacement and the contact stress. However, the energy decreases rapidly for large time steps. Figure 4 shows that for a well-posed method, the energy tends to be conserved for small time steps, but Figures 5 and 6 show that, with an ill-posed method (such as classical discretizations), there is an energy loss at the impact which does not vanish when the time step and the mesh size decrease. This means that with an ill-posed method, we do not approximate a physical solution of the problem whenever one expects energy conservation to be satisfied at the limit.

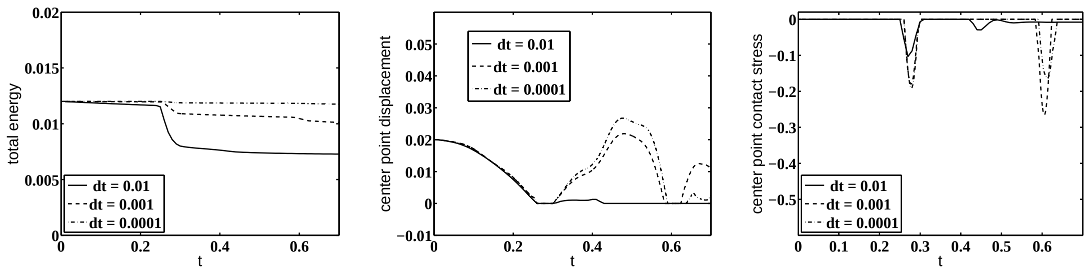

Figure 4: *Energy evolution (left), Displacement at the center point* $(0.5, 0.5)$ *(center) and Contact stress at the center point (right) -* P_1+/P_0 *method with a backward Euler scheme and* $h = 0.1$.

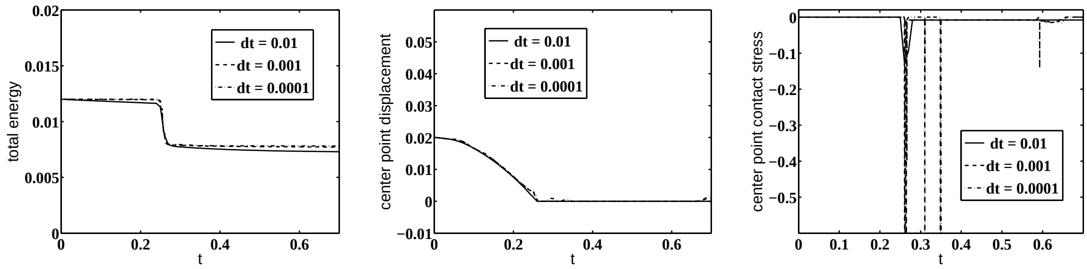

Figure 5: *Energy evolution (left), Displacement at the center point* $(0.5, 0.5)$ *(center) and Contact stress at the center point (right) -* P_1/P_0 *method with a backward Euler scheme and* $h = 0.1$.

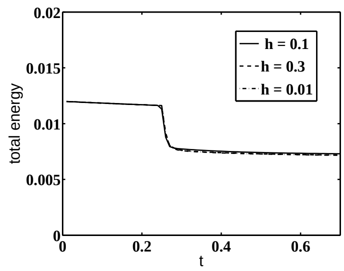

Figure 6: *Energy evolution for a* P_1/P_0 *method, a backward Euler scheme and* $\Delta t = 0.001$*, for different values of the mesh size.*

5.3 Case of a plate

A steel rectangular panel is considered of length $L = 120\ cm$, width $l = 40\ cm$ and thickness $\varepsilon = 0.5\ cm$. It means domain Ω is $]0, L[\ \times\ ]0, l[$. The flexural rigidity is $D = 1.923\ 10^4$, corresponding to $E = 210\ GPa$ and $\nu = 0.3$, while $\rho = 7.77\ 10^3 kg/m^3$. This plate is clamped along one edge and free along the three others. Moreover, only the following kind of obstacle will be considered here. It is a flat obstacle under the whole plate, which reads

$$g_2(x_1, x_2) = +\infty \quad , \quad g_1(x_1, x_2) = -0.1 \quad , \quad \forall\ (x_1, x_2) \in \Omega \quad .$$

Finally, as we are mainly interested to study energy conservation, we consider two cases where there is no loading $f(x, t) \equiv 0$ for all x and t. In a first case, the plate is clamped and all energy is contained in an initial displacement u_0 , obtained as the static equilibrium of the plate under a constant load $f_0 = 14600\ N$ and an initial velocity $v_0 = 0$. In a second case, the plate is free and all energy is contained in an initial velocity $v_0 = 1\ m/s.$, and the initial displacement $u_0 = 0$.

Figures 7, 8 and 9 present the energy evolution for different time steps in the case of the clamped plate, for the midpoint scheme and for the different finite element discretizations. We notice that energy is oscillating with a global increase for the larger time step and tends to be conserved when the time step decreases.

The energy evolution for the free plate is represented in Figures 10, 11 and 12. Here again, the energy is oscillating but with no global increase and tends also to be conserved when the time step decreases.

Note that the numericaly observed stability condition seems to be more restrictive for triangles than for quadrilaterals.

The deflection at free corners for the free plate is represented in Figures 13 and 14. When the time step decreases, the mean velocity after impact tends to be close to the (opposite of the) velocity before impact. However, some slight oscillations can be noted which correspond to some energy transfer from the vertical rigid body motion to an oscillatory eigenmode of the structure.

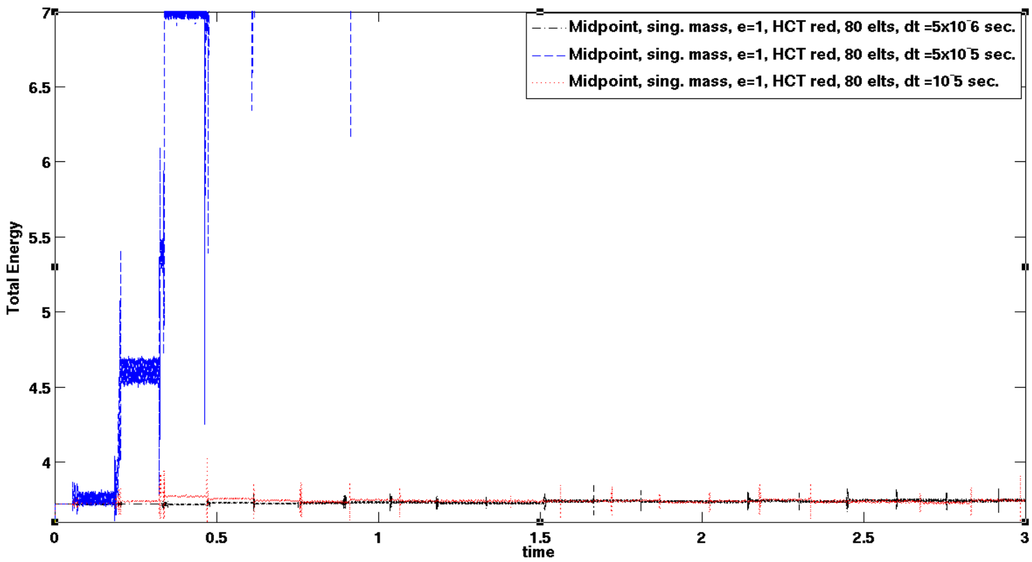

Figure 7: Clamped plate. Energy for different time steps. Reduced HCT , 80 triangles. Midpoint scheme.

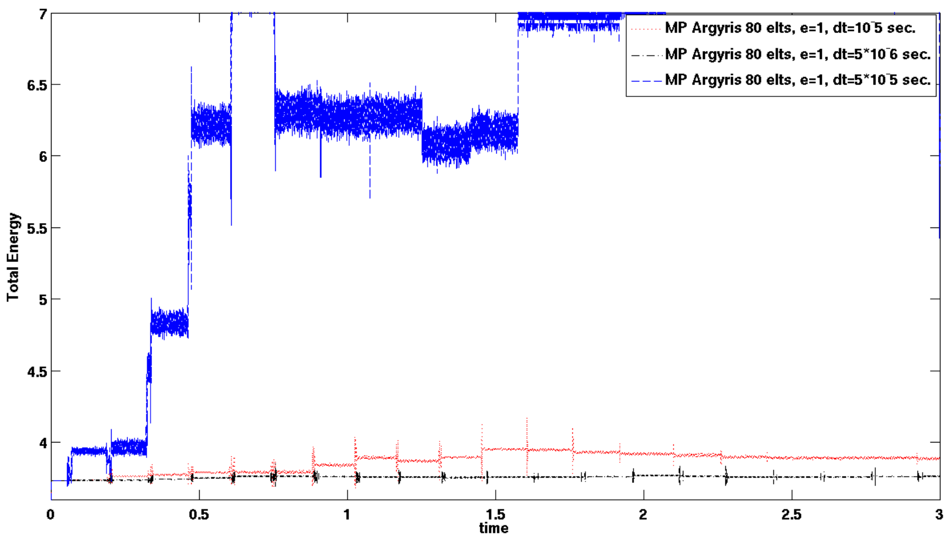

Figure 8: Clamped plate. Energy for different time steps. Argyris , 80 triangles. Midpoint scheme.

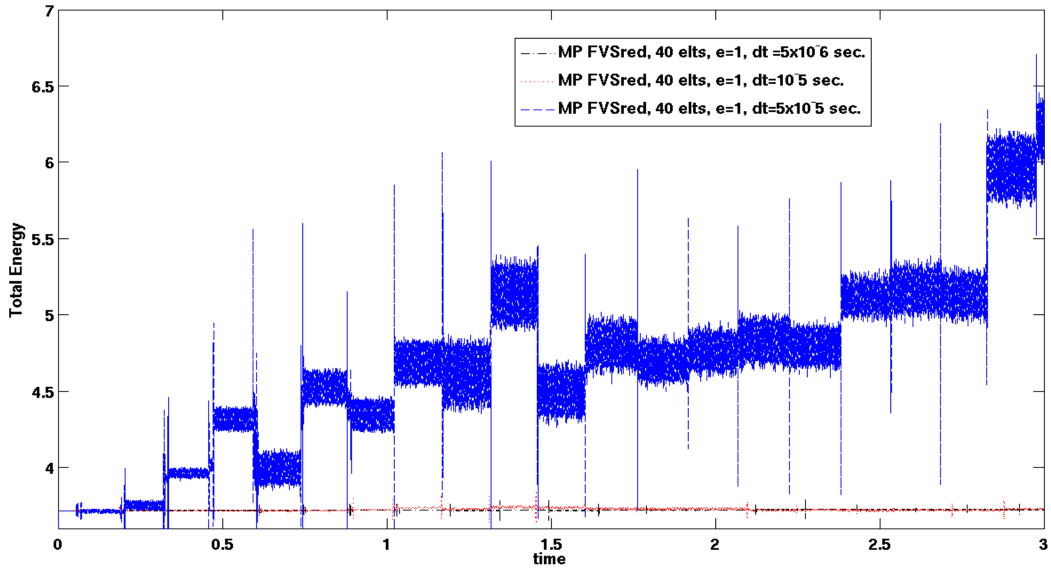

Figure 9: Clamped plate. Energy for different time steps. Reduced FVS , 40 quadrilaterals. Midpoint scheme.

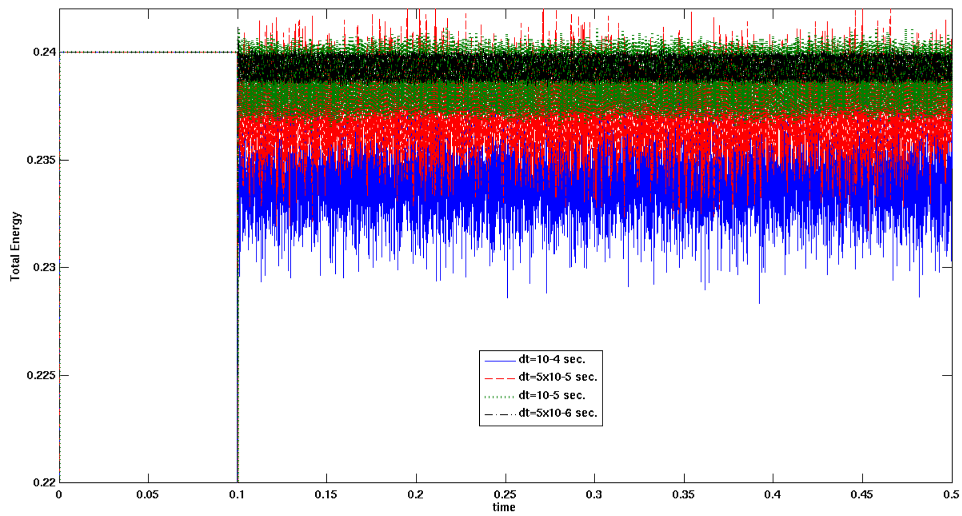

Figure 10: Free plate. Energy for different time steps. Reduced HCT , 80 triangles. Midpoint scheme.

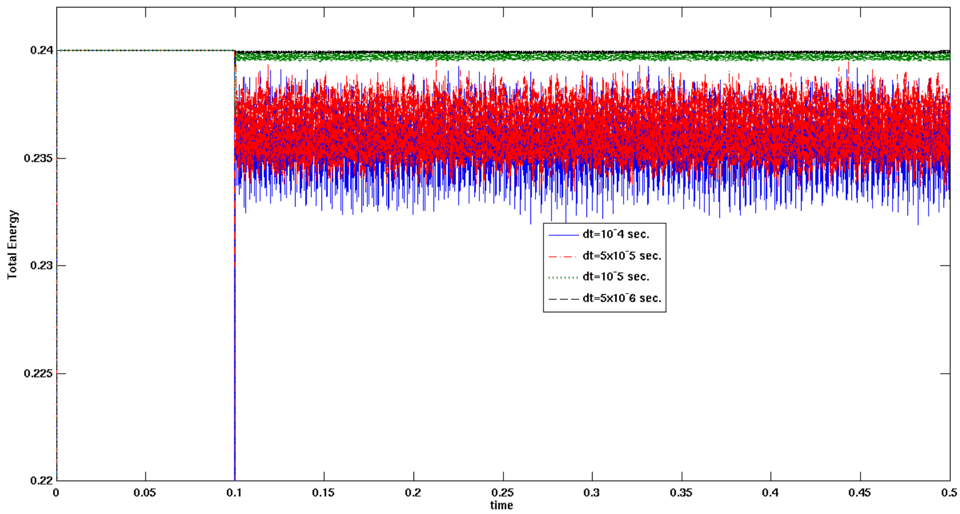

Figure 11: Free plate. Energy for different time steps. Argyris , 80 triangles. Midpoint scheme.

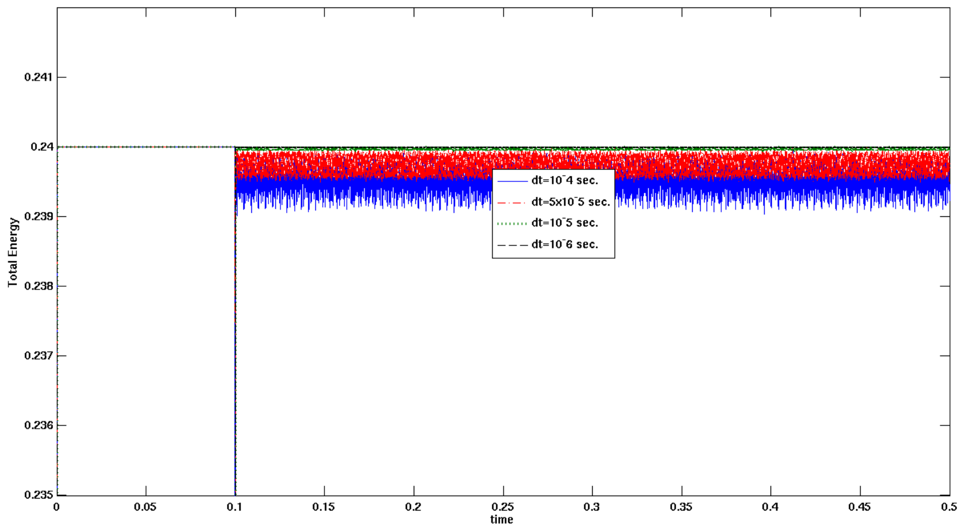

Figure 12: Free plate. Energy for different time steps. Reduced FVS , 40 quadrilaterals. Midpoint scheme.

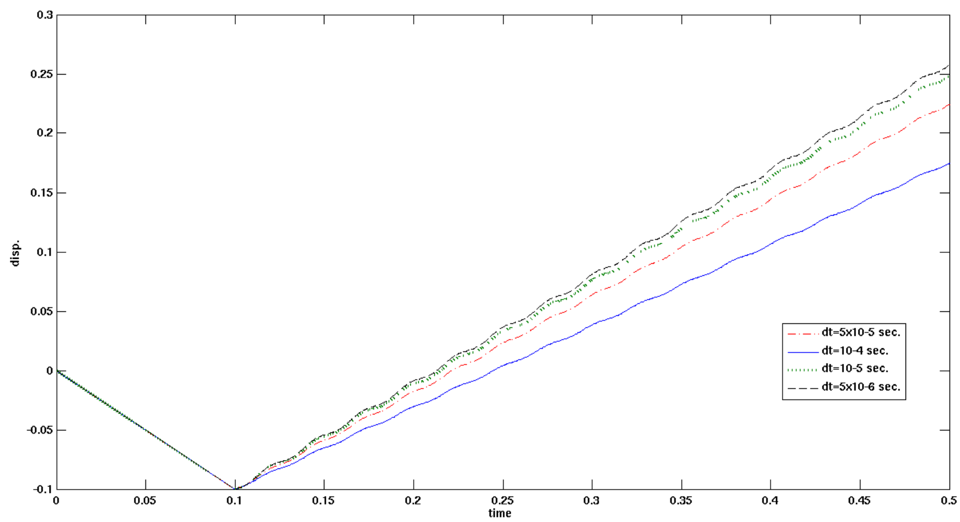

Figure 13: Free plate. Deflection at a free corner for different time steps. Reduced HCT , 80 triangles. Midpoint scheme.

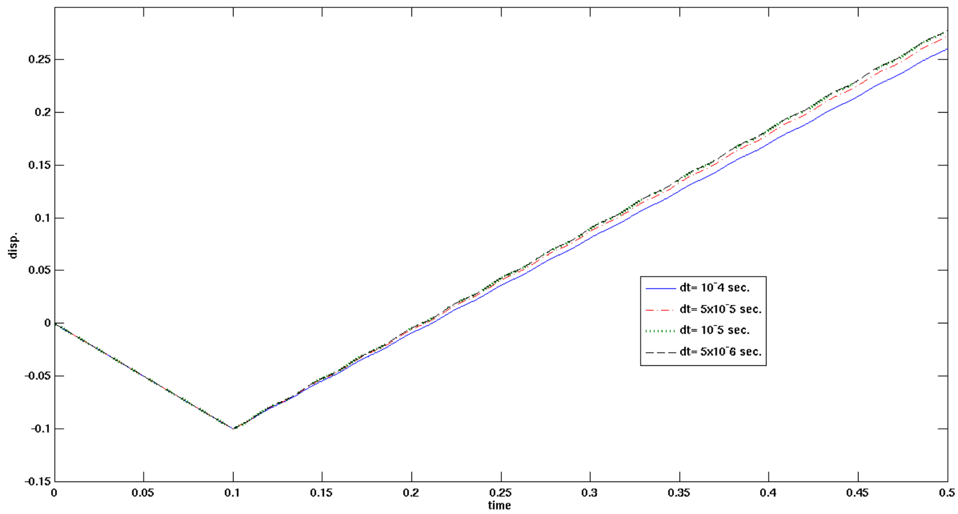

Figure 14: Free plate. Deflection at a free corner for different time steps. Argyris , 80 triangles. Midpoint scheme.

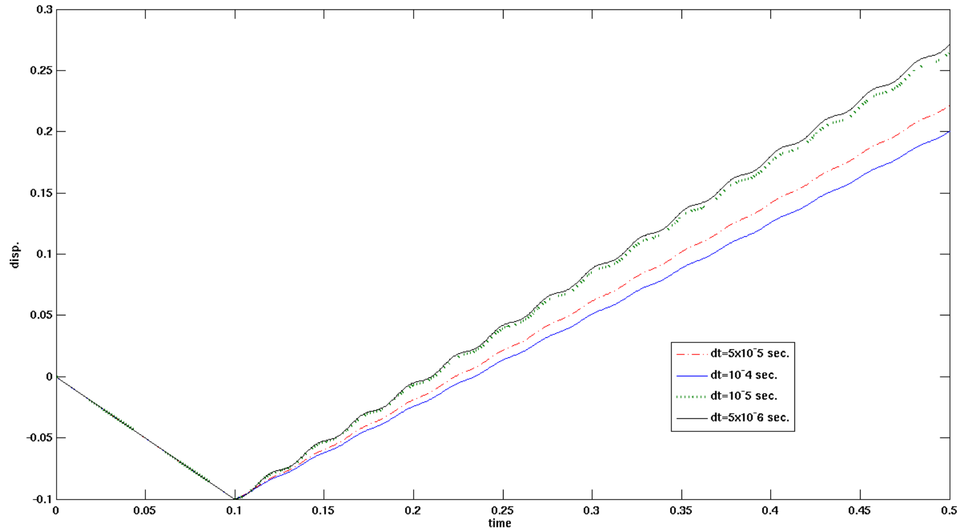

Figure 15: Free plate. Deflection at a free corner for different time steps. Reduced FVS , 40 quadrilaterals. Midpoint scheme.

6 Open problems and perspectives

The semi-discretization obtained by the singular dynamic method leads to a problem which is equivalent to a regular Lipschitz ordinary differential equation (see also [28] for a slightly different approach). This method generalizes in a sense the ones presented in [12, 6] with the advantage that no artificial modification of the mass matrix is necessary.

This is compared to the classical semi-discretizations, for example with finite element methods, which give a problem in time which is a measure differential inclusion (see [18, 19, 20, 21]). Such a differential inclusion is systematically ill-posed, unless an additional impact law is considered.

Concerning thin structures, as it is illustrated by Figures 4, 5 and 6, numerical schemes do not necessarily converge toward the same solution. The limit solution may have different characteristics of impact energy loss. This suggests that in the case of thin structures, modeling of the restitution of the impact energy should be added to the impact law. The proposed semi-discretization being conservative in energy, it corresponds to a total restitution of the impact energy. A classical semi-discretization by finite elements with an implicit

Euler scheme, as in Figure 6 corresponds to a certain loss of impact energy. Note that this energy loss is not necessarily maximal. Finally, the dissipation of the impact will certainly depend at the same time on the type of semi-discretization in space, the type of time integration scheme, the ratio between the space step and the time step, the kind of discretization of the contact conditions and finally of how the structure impacts the thin rigid obstacle (more or less obliquely, for instance). For the moment, the accurate modeling of the energy restitution at impact for the approximation of the dynamics of thin structures seems a little studied area. An interesting perspective is to try to characterize the different numerical schemes according to their characteristic in term of energy restitution. Other fields to investigate are for example the influence of the sigular mass method on coupled models of flexion and membrane (traction/compression), or on the Mindlin-Reissner plate model.

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